NIEMEIER LATTICES AND K3 GROUPS

Dedicated to Professor I. Dolgachev on the occassion of his sixtieth birthday

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Abstract. In this note, we consider $K3$ surfaces X with an action by the alternating group A_5. We show that if a cyclic extension $A_5.C_n$ acts on X then $n = 1$, 2, or 4. We also determine the A_5-invariant sublattice of the K3 lattice and its discriminant form.

Introduction

We work over the complex numbers field $\mathbf{C}$. A **K3** surface X is a simply connected projective surface with a nowhere vanishing holomorphic 2-form ω_X. In this note, we will consider finite groups in $\mathrm{Aut}(X)$. An element $h \in \mathrm{Aut}(X)$ is **symplectic** if h acts trivially on the 2-form ω_X. A group $G_N \subseteq \mathrm{Aut}(X)$ is **symplectic** if every element of G_N is symplectic.

According to Nikulin [Ni1], Mukai [Mu1] and Xiao [Xi], there are exactly 80 abstract finite groups which can act symplectically on $K3$ surfaces. Among these 80, there are exactly three non-abelain simple groups A_5, $L_2(7)$ and A_6.

To be more precise, as in **(1.0)** below, for every finite group G acting on a $K3$ surface X, the symplectic elements of G (i.e., those h acting trivially on the non-zero 2-form ω_X) form a normal subgroup G_N such that $G/G_N \cong \mu_I$ (the cyclic group of order I in $\mathbf{C}^*$). Namely, we have $G = G_N.\mu_I$ (see **Notation** below). The natural number $I = I(G)$ is determined by G and called the **transcendental value** of G.

It is proved in [OZ3] and [KOZ1, 2] that when G_N is either one of the two bigger simple groups above, the transcendental value $I(G) \neq 3$. As expected or unexpected, the same is

true for the smaller (indeed the smallest non-abelian simple group) A_5:

Theorem A. Suppose that $G = A_5.\mu_I$ acts faithfully on a $K3$ surface (assuming that $G_N = A_5$). Then $G = A_5 : \mu_I$ (semi-direct product) and $I = 1$, 2, or 4.

In general, for a group of the form $G = A_5.C_n$ acting on a $K3$ surface (here G_N might be bigger than A_5; and C_n an abstract cyclic group of order n), we have a similar result:

Theorem B. Suppose that a group of the form $G = A_5.C_n$ acts faithfully on a $K3$ surface. Then $G = A_5 : C_n$ and $I = 1$, 2, or 4. Moreover, $G_N = A_5$ (and hence $C_n = \mu_n$ in the notation above or **(1.0)**) unless $G = G_N = S_5$.

We can determine the A_5-invariant sublattice of the $K3$ lattice in the result below, which has application in helping determine the transcendental lattice T_X and hence the surface itself (when rank $T_X = 2$).

Theorem C. Suppose that A_5 acts faithfully on a $K3$ surface X. Then we have:
(1) The A_5-invariant sublattice L^{A_5} of the $K3$ lattice $L = H^2(X, \mathbf{Z})$ has rank 4. The A_5-invariant sublattice $S_X^{A_5}$ of the Neron Severi lattice S_X has rank equal to 1 or 2.
(2) The discriminant group $A_{L^{A_5}} = \mathrm{Hom}(L^{A_5}, \mathbf{Z})/L^{A_5}$ equals one of the following (see Theorem **(2.1)** for the corresponding intersection forms):

$$\mathbf{Z}/(30) \oplus \mathbf{Z}/(30), \quad \mathbf{Z}/(30) \oplus \mathbf{Z}/(10), \quad \mathbf{Z}/(20) \oplus \mathbf{Z}/(20),$$
$$\mathbf{Z}/(60) \oplus \mathbf{Z}/(20), \quad \mathbf{Z}/(60) \oplus \mathbf{Z}/(20) \oplus \mathbf{Z}/(2) \oplus \mathbf{Z}/(2).$$

Remark D.
(1) In [Z2], it is proved that there is no faithful action of $A_5.\mu_4$ on a $K3$ surface. So the I in Theorems A and B can only be 1 or 2.
(2) Theorem C is used in [Z2, Lemma 3.5]. The proofs of Theorems A, B and C here are independent of the paper [Z2].
(3) Theorem C is applicable in the following situation: Suppose in addition that a non-symplectic involution $g \in \mathrm{Aut}(X)$ commutes with every element in A_5 and that the fixed

locus X^g is a union of a genus ≥ 2 curve C and s (≥ 1) smooth rational curves D_i. Then $S_X^{A_5}$ contains $L_0 = \mathbf{Z}[C, \sum_{i=1}^{s} D_i]$ as a sublattice of finite index d_1. Note that L^{A_5} contains $S_X^{A_5} \oplus T_X$ as a sublattice of finite index d. So $|L_0||T_X| = d_1^2|S_X^{A_5}||T_X| = d_1^2 d^2 |L^{A_5}|$ while $-|L^{A_5}| = 30^2, 3 \times 10^2, 20^2$, 3×20^2, or 3×40^2 as given in Theorem C. This is a restriction imposed on $|T_X|$. In [Z2], we determined d_1, d, and $|T_X|$ using the existence of the extra μ_4 in (the impossible case:) $A_5.\mu_4$ where T_X then has the intersection form diag$[2m, 2m]$ for some $m \geq 1$.

(4) The same construction in [OZ3, Appendix] shows that there is a smooth non-isotrivial family of $K3$ surfaces $f : \mathcal{X} \to \mathbf{P}^1$ such that all fibres admit A_6 actions and infinitely many of them are algebraic K3 surfaces. So, the symplectic part alone can not determine the surface uniquely, and the study of transcendental value is needed.

The main tools of the paper are the Lefschetz fixed point formula (both the topological version and vector bundle version due to Atiyah-Segal-Singer [AS2, 3]), the representation theory on the $K3$ lattice and the study on automorphism groups of Niemeier lattices (in connection with Golay binary or ternary codes) where the latter is much inspired by Conway-Sloane [CS], Kondo [Ko1] and Mukai [Mu2].

We believe that the way of combining different very powerful machinaries to deduce results as done in the paper should be applicable to the study of other problems. Our humble paper also demonstrates the powerfulness and depth of these algebraic results in the study of geometry.

Notation.

1. S_n is the symmetric group in n letters, A_n ($n \geq 3$) the alternating group in n letters, $\mu_I = \langle \exp(2\pi\sqrt{-1})/I \rangle$ the multiplicative group of order I in $\mathbf{C}^*$ and C_n an abstract cyclic group of order n.

2. For a group G, we write $G = A.B$ if A is normal in G so that $G/A = B$. We write $G = A : B$ if assume further that A is normal in G and B is a subgroup of G such that the composition $B \to G \to G/A = B$ is the identity (we say then that G is a **semi-direct product** of A and B).

3. For groups $H \leq G$ and $x \in G$ we denote by $c_x : H \to G$ ($h \mapsto c_x(h) = x^{-1}hx$) the **conjugation** map.

4. For a $K3$ surface X, we let S_X and T_X be the Neron-Severi and transcendental lattices. So the $K3$ lattice $H^2(X, \mathbf{Z})$ contains $S_X \oplus T_X$ as a sublattice of finite index.

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§1. Preliminary Results

(1.0). In this section, we will prepare some basic results to be used late. Let X be a $K3$ surface with a non-zero 2-form ω_X and let $G \subseteq \mathrm{Aut}(X)$ be a finite group of automorphisms. For every $h \in G$, we have $h^*\omega_X = \alpha(h)\omega_X$ for some scalar $\alpha(h) \in \mathbf{C}^*$. Clearly, $\alpha : G \to \mathbf{C}^*$ is a homomorphism. A fact in basic group theory says that $\alpha(G)$ is a finite cyclic group, so $\alpha(G) = \mu_I = \langle \exp(2\pi\sqrt{-1}/I) \rangle$ for some $I \geq 1$. This natural number $I = I(G)$ is called the **transcendental** value of G. It is known that $I = I(G)$ for some G if and only if that the Euler function $\varphi(I) \leq 21$ and $I \neq 60$ [MO].

Set $G_N = \mathrm{Ker}(\alpha)$. Then we have the **basic exact sequence** below:

$$1 \longrightarrow G_N \longrightarrow G \xrightarrow{\alpha} \mu_I \longrightarrow 1.$$

For the G in the basic exact sequence, we write $G = G_N.\mu_I$, though there is no guarantee that $G = G_N : \mu_I$ (a semi-direct product).

1.0A. If G is a finite perfect group, i.e., the commutator group $[G, G] = G$ (especially, if G is a non-abelian simple group like A_5), then $G = G_N$.

1.0B. G_N acts trivially on the transcendental lattice T_X (Lefschetz theorem on $(1,1)$-classes).

1.0C. If a subgroup $H \le G_N$ fixes a point P, then $H < SL(T_{X,P}) \cong SL_2(\mathbf{C})$ [Mu1, **(1.5)**]. The finite subgroups of $SL_2(\mathbf{C})$ are listed up in [Mu1, **(1.6)**]. These are cyclic C_n, binary dihedral (or quaternion) Q_{4n} ($n \ge 2$), binary tetrahedral T_{24}, binary octahedral O_{48} and binary icosahedral I_{120}.

Lemma 1.1. *Suppose that $G := A_5.\mu_I$ acts faithfully on a $K3$ surface X.*

(1) The Picard number $\rho(X) \ge 19$, and $I = 1, 2, 3, 4, 6$. Moreover, $\rho(X) = 20$ if $I \ge 3$.

(2) We have $G = A_5 : \mu_I$, i.e., a semi-prodcut of a normal subgroup A_5 and a subgroup μ_I of G. Moreover, $G = A_5 \times \mu_I$ if $I = 3$.

Proof. (1) In notation of [Xi, the list], $\rho(X) = \operatorname{rank} S_X \ge c + 1 = 19$. Also the Euler function $\varphi(I)$ divides $\operatorname{rank} T_X = 22 - \rho(X)$ by [Ni1, Theorem 0.1]. So (1) follows.

(2) Let $g \in G$ such that $\alpha(g)$ is a generator of μ_I. Since $\mathrm{Aut}(A_5) = S_5 > A_5$ and the conjugation homomorphism $A_5 \to \mathrm{Aut}(A_5)$ ($x \mapsto c_x$) is an isomorphism onto A_5, the conjugation map c_g equals $c_{(12)a}$ or c_a on A_5 for some $a \in A$. Replacing g by ga^{-1}, we may assume that $c_g = c_{(12)}$ or c_{id}. Thus g^2 commutes with every element in A_5. If $2|I$, then $g^I \in \mathrm{Ker}(\alpha) = A_5$ is in the centre of A_5 (which is trivial) and hence $\mathrm{ord}(g) = I$; thus $G = A_5 : \mu_I$. If $I = 3$, then $\gcd(3, \mathrm{ord}(g)/3) = 1$ as proved in [IOZ] or [Og, Proposition 5.1]; so replacing g by g^ℓ with $\ell = \mathrm{ord}(g)/3$ (or $2\mathrm{ord}(g)/3$), we have $G = A_5 \times \langle g \rangle = A_5 \times \mu_3$.

The second result below [Ni1, §5] is crucial in classifying symplectic groups in [Mu1]. For the first, see [Ni2], [Z1] or [Z2, Lemma 1.2], where the Hodge index theorem is also used here.

Lemma 1.2. *(1) Let h be a non-symplectic involution on a $K3$ surface X. Then X^h is a disjoint union of s smooth curves C_i with $0 \le s \le 10$. To be precise, X^h (if not empty) is either a disjoint union of a genus ≥ 2 curve C and a few $\mathbf{P}^1$'s, or a disjoint union of a few elliptic curves and $\mathbf{P}^1$'s, or a disjoint union of a few $\mathbf{P}^1$'s.*

(2) If δ is a non-trivial symplectic automorphism of finite order on a $K3$ surface X, then $\mathrm{ord}(\delta) \le 8$ and X^δ is a finite set. To be precise, if $\mathrm{ord}(\delta) = 2, 3, 4, 5, 6, 7, 8$, then $|X^\delta| = 8, 6, 4, 4, 2, 3, 2$, respectively. In particular, if $A_5 \subseteq \mathrm{Aut}(X)$ then $\sum_{\delta \in A_5} \chi_{\mathrm{top}}(X^\delta) = 360$

(see **(1.0A)**).

For an automorphism h on a smooth algebraic surface Y, we split the pointwise fixed locus as the disjoint union of 1-dimensional part and the isolated part: $Y^h = Y^h_{1-\mathrm{dim}} \coprod Y^h_{\mathrm{isol}}$. The proof of (1) below is similar to that for (1) in **(1.2)**.

1.3. (1) $Y^h_{1-\mathrm{dim}}$ (if not empty) is a disjoint union of smooth curves.

(2) The Euler number $\chi_{\mathrm{top}}(Y^h_{1-\mathrm{dim}}) = \sum_C (2 - 2g(C)) = 2n_h$ for some integer n_h, where C runs in the set $Y^h_{1-\mathrm{dim}}$ of curves.

(3) The Euler number $\chi_{\mathrm{top}}(Y^h) = m_h + 2n_h$, where $m_h = |Y^h_{\mathrm{isol}}|$.

The results of [IOZ] below follow from the application of Lefschetz fixed point formula to the trivial vector bundle in Atiyah-Segal-Singer [AS2, AS3, pages 542 and 567]. For a proof, see [OZ1, Lemma 2.3] and [Z2, Proposition 1.4].

Lemma 1.4. Let X be a $K3$ surface and let $h \in \mathrm{Aut}(X)$ be of order I such that $h^*\omega_X = \eta_I \omega_X$ for some primitive I-th root η_I of 1.

(1) Suppose that $I = 3$. Then $m_h = 3 + n_h$ and hence $\chi_{\mathrm{top}}(X^h) = 3(1 + n_h)$. Moreover, $-3 \le n_h \le 6$.

(2) Suppose that $I = 3$. If $\delta \in \mathrm{Aut}(X)$ is symplectic of order 5 and commutes with h. Then $|X^{h\delta}| = 4$.

The following result can be found in [Ni1, Theorem 0.1], [MO, Lemma (1.1)], or [OZ3, Lemma (2.8)].

Lemma 1.5. Suppose that X is a $K3$ surface of Picard number $\rho(X) = 20$ and g an order-3 automorphism such that $g^*\omega_X = \eta_3 \omega_X$ with a primitive 3rd root η_3 of 1. Then we can express the transcendental lattice T_X as $T_X = \mathbf{Z}[t_1, t_2]$ so that $t_2 = g^*(t_1)$, $g^*(t_2) = -(t_1 + t_2)$. In particular, for some $m \ge 1$, the intersection form $(t_i.t_j) = \begin{pmatrix} 2m & -m \\ -m & 2m \end{pmatrix}$.

Now we assume that $G = G_N.\mu_I$ (with $I = I(G)$) acts on a $K3$ surface X. When $G_N = A_5$, we will determine the action of G_N on the Neron Severi lattice S_X of X:

Lemma 1.6. (1) Suppose that A_5 acts on a $K3$ surface X, and $\operatorname{rank} S_X = 20$ (this is true if $I \geq 3$ by **(1.1)**). Then we have the irreducible decomposition below (in the notation of Atlas for the characters of A_5), where χ_1 (the trivial character), χ_4 and χ_5 have dimensions 1, 4 and 5, respectively, where χ_i' is a copy of χ_i:

$$S_X \otimes \mathbf{C} = \chi_1 \oplus \chi_1' \oplus \chi_4 \oplus \chi_4' \oplus \chi_5 \oplus \chi_5'.$$

(2) For conjugacy class nA (and nB) of order n in A_5 and the characters χ_i of A_5, we have the following **Table 1** from [Atlas], where Z is respectively $1A$, $2A$, $3A$, $5A$ or $5B$:

$$[\chi_1, \chi_2, \chi_3, \chi_4, \chi_5](Z) = [1, 3, 3, 4, 5], \quad [1, -1, -1, 0, 1], \quad [1, 0, 0, 1, -1],$$
$$[1,\ (1-\sqrt{5})/2,\ (1+\sqrt{5})/2,\ -1, 0], \quad [1,\ (1+\sqrt{5})/2,\ (1-\sqrt{5})/2,\ -1, 0].$$

Proof. The assertion(1) appeared in [Z2]. For the readers' convenience, we reprove it here. Applying the Lefschetz fixed point formula to the action of A_5 on $H^*(X, \mathbf{Z}) = \oplus_{i=0}^4 H^i(X, \mathbf{Z})$ and noting that $H^2(X, \mathbf{Z})$ contains $S_X \oplus T_X$ as a finite index sublattice, we obtain (see also **(1.0A-B)** and **(1.2)**):

$$2 + \operatorname{rank} T_X + \operatorname{rank}(S_X)^{A_5} = \operatorname{rank} H^*(X, \mathbf{Z})^{A_5} = \frac{1}{|A_5|} \sum_{a \in A_5} \chi_{\text{top}}(X^a) = 360/60 = 6.$$

Thus $\operatorname{rank} S_X^{A_5} = 2$. So the irreducible decomposition is of the following form, where a_i are non-negative integers:

$$S(X) \otimes \mathbf{C} = 2\chi_1 \oplus a_2\chi_2 \oplus a_3\chi_3 \oplus a_4\chi_4 \oplus a_5\chi_5.$$

Using the topological Lefschetz fixed point formula, the fact that $\operatorname{rank} T(X) = 2$ and **(1.0B)**, we have, for $a \in A_5$, that:

$$\chi_{\text{top}}(X^a) = 2 + \operatorname{rank} T_X + \operatorname{Tr}(a^* | S(X))$$

Running a through the five conjugacy classes and calculating both sides, using **(1.2)** and

the character Table 1 in (2), we obtain the following system of equations:

$$20 = 2 + 3(a_2 + a_3) + 4a_4 + 5a_5,$$

$$4 = 2 - (a_2 + a_3) + a_5,$$

$$2 = 2 + a_4 - a_5,$$

$$0 = 2 + \frac{1-\sqrt{5}}{2}a_2 + \frac{1+\sqrt{5}}{2}a_3 - a_4,$$

$$0 = 2 + \frac{1+\sqrt{5}}{2}a_2 + \frac{1-\sqrt{5}}{2}a_3 - a_4.$$

Now, we get the result by solving this system of Diophantine equations.

The two results below are either easy or well known and will be frequently used in the arguments of the subsequent sections.

Lemma 1.7. Let $f : A_5 \to S_r$ $(r \geq 2)$ be a homomorphism.
(1) If $r = 2$, 3, or 4, then f is trivial.
(2) If $\mathrm{Im}(f)$ is a transitive subgroup of the full symmetry group S_r in r letters $\{1, 2, \ldots, r\}$ (whence $r \geq 5$ by (1)), then $r||A_5|$ with $|A_5|/r$ equal to the order of the subgroup of A_5 stabilizing the letter 1, so $r \in \{5, 6, 10, 12, 15, 20, 30\}$.

Lemma 1.8. (1) $\mathrm{Aut}(\mathbf{P}^1)$ includes A_5 but not S_5 [Su, Theorem 6.17]. The action of A_5 on $\mathbf{P}^1$ is unique up to isomorphisms.
(2) Every A_5 in $PGL_3(\mathbf{C})$ is the image of an A_5 in $SL_3(\mathbf{C})$.
(3) The conjugation by $(12) \in S_5$ switches the two 3-dimensional characters χ_2 and χ_3 of A_5 [Atlas].
(4) If $\mathrm{id} \neq f \in \mathrm{Aut}(\mathbf{P}^1)$ is an automorphism of finite order, then f fixes exactly two point of $\mathbf{P}^1$ (by the diagonalization of a lifting of f to $GL_2(\mathbf{C})$).
(5) If f_r $(r = 2$ or $3)$ is an order$-r$ automorphism of an elliptic curve E, then either f_r acts freely on E, or the fix locus satisfies $|X^{f_r}| = 4$ (resp. $= 3$) if $r = 2$ (resp. $r = 3$) (by the Hurwitz formula).

Proof. (1) For the uniqueness of the action of A_5 on $\mathbf{P}^1$, one may assume the representation

of $D_{10} = \langle \gamma = (12345), \sigma = (14)(23) \rangle$ is given by $\gamma : z \to \eta z$ with η a primitive 5-th root of 1 and $\sigma : z \to \alpha/z$. Note that $A_5 = \langle \gamma, \varepsilon \rangle$ with $\varepsilon = (12)(34)$. If one lets $\varepsilon : z \to (az+b)/(cz+d)$ be in $\mathrm{Aut}(\mathbf{P}^1)$, then one can check that $d = -a$ because $\mathrm{ord}(\varepsilon) = 2$, and also $b = -c\alpha$ because ε commutes with σ. So $\varepsilon : z \to (z - \alpha e)/(ez - 1)$ with $e = c/a$. The commutativity of $\varepsilon\sigma\gamma^2\varepsilon = (12)(45)$ with $\sigma\gamma^{-1} = (15)(24)$ implies that $e^2\alpha = \eta + \eta^{-1} - 1$. Now let $\rho : z \to e\alpha/z$ be in $\mathrm{Aut}(\mathbf{P}^1)$. Then $\rho^{-1}\gamma\rho : z \to \eta^{-1}z$, $\rho^{-1}\sigma\rho : z \to e^2\alpha/z$ and $\rho^{-1}\varepsilon\rho : z \to (z - e^2\alpha)/(z-1)$. Hence the action of A_5 on $\mathbf{P}^1$ is unique modulo isomorphisms.

(2) For an A_5 in $SL_3(\mathbf{C})$, see [Bu, §232]. The inverse $\widetilde{A}_5 \subset SL_3(\mathbf{C})$ of an $A_5 \subset PGL_3(\mathbf{C})$ is of the form $\widetilde{A}_5 = A_5 : \mu_3$ (indeed, a direct product) because the Schur multiplier $M(A_5) = 2$, coprime to 3 [Atlas]. So (2) follows.

§2. Alternating groups actions on the Niemeier lattices

For a $K3$ surface X, denote by $L = H^2(X, \mathbf{Z})$ the **K3 lattice**, $S_X = \mathrm{Pic}\, X$ (now) the **Neron-Severi lattice** and T_X the **transcendental lattice**. So $T_X = S_X^\perp$ in L and L contains a finite-index sublattice $S_X \oplus T_X$.

(2.0). Suppose that $G_N = A_5$ acts faithfully on X. In this section we shall prove Theorem C which is part of **(2.1)** below. Indeed, by the proof of **(1.6)**, we have $\mathrm{rank}\, L^{G_N} = \mathrm{rank}\, T_X + \mathrm{rank}\, S_X^{G_N} = 4$, so $(\mathrm{rank}\, T_X, \mathrm{rank}\, S_X, \mathrm{rank}\, S_X^{G_N}) = (2, 20, 2)$ or $(3, 19, 1)$.

Denote by $L^{G_N} := \{x \in L \mid g^*x = x \text{ for all } g \in G_N\}$ and its orthogonal $L_{G_N} := (L^{G_N})^\perp = \{x \in L \mid (x, y) = 0 \text{ for all } y \in L^{G_N}\}$. Then L^{G_N} contains $S_X^{G_N} \oplus T_X$ as a sublattice of finite index by **(1.0A-B)**.

By [Ko1, Lemmas 5 and 6], there are a (non-Leech) Niemeier lattice $N(Rt)$, a primitive embedding $A_1 \oplus L_{G_N} \subset N(Rt)$ and a faithful action of G_N on $N(Rt)$ such that $L_{G_N} = N(Rt)_{G_N}$, and the action of G_N on the summand A_1 is trivial and stabilizes a Weyl chamber (one of whose codimension one faces corresponds to this A_1). Moreover, $G_N \le S(N(Rt)) := O(N(Rt))/W(N(Rt)) \le O(Rt)/W(N(Rt))(=: \mathrm{Sym}(Rt))$, where $\mathrm{Sym}(Rt)$ is

the full symmetry group of the Coxeter-Dynkin diagram Rt. Note that $\mathrm{rank}\, N(Rt)^{G_N} = 2 + \mathrm{rank}\, L^{G_N} = 6$ and the discriminant groups satisfy:

$$(*) \quad A_{L^{G_N}} \cong A_{L_{G_N}}(-1) = A_{N(Rt)_{G_N}}(-1) \cong A_{N(Rt)^{G_N}}.$$

Now $N(Rt)^{G_N}$ is a rank 6 lattice generated by $e_1, \ldots, e_6$ say. Denote by $M = (e_i.e_j)$ the intersection matrix and $M^{-1} = (f_1, \ldots, f_6)$ with f_j column vectors and set $e_i^* = (e_1, \ldots, e_6)f_i$. Then $(N(Rt)^{G_N})^\vee = \mathrm{Hom}(N(Rt)^{G_N}, \mathbf{Z})$ has the dual basis $\{e_1^*, \ldots, e_6^*\}$ with the intersection matrix $(e_i^*.e_j^*)_{1 \le i,j \le 6} = M^{-1}$. The discriminant groups satisfy $A_{L^{G_N}} \cong A_{N(Rt)^{G_N}} = \mathbf{Z}[e_1^*, \ldots, e_6^*]/\mathbf{Z}[e_1, \ldots, e_6]$.

In this section, we shall prove the following result (much inspired by [Ko1]), which (and the proof of which) should be useful in studying $\mathrm{Aut}(X)$ from the $K3$ lattice point of view. This result is used in [Z2, Lemma 3.5].

Theorem 2.1. Suppose that $G_N = A_5$ acts faithfully on a $K3$ surface X.

(1) We have $Rt = 24A_1$ or $Rt = 12A_2$. The lattice $N(Rt)^{G_N}$ is of rank 6 and generated by $e_1, \ldots, e_6$ say. Denote by $M = (e_i.e_j)$ the intersection matrix and write $M^{-1} = (f_1, \ldots, f_6)$ with f_j column vectors and set $e_i^* = (e_1, \ldots, e_6)f_i$.

(2) If $Rt = 24A_1$, then the orbit decomposition of the G_N-action on the 24 simple roots is either one of

$$\text{(i) } [1,1,5,5,6,6], \quad \text{(ii) } [1,1,1,5,6,10], \quad \text{(iii) } [1,1,1,1,5,15], \quad \text{(iv) } [1,1,1,1,10,10].$$

If $Rt = 12A_2$, then the orbit decomposition of the G_N-action on the 24 simple roots is either one of

$$(v)\ [1,1,1,1,10,10], \quad (vi)\ [1,1,5,5,6,6],$$

where in (v) (resp. (vi)) $10A_2$ (resp. $5A_2$, or $6A_2$) is split into two orbits with 10 (resp. 5, or 6) disjoint roots each.

(3) For Case(2i), the intersection matrix $M_1 = (e_i.e_j)$ and its inverse M_1^{-1} are respectively:

$$\begin{pmatrix} -2 & 0 & 0 & -1 & -1 & -1 \\ 0 & -2 & 0 & -1 & -1 & -1 \\ 0 & 0 & -10 & 0 & 0 & -5 \\ -1 & -1 & 0 & -4 & -1 & -1 \\ -1 & -1 & 0 & -1 & -4 & -1 \\ -1 & -1 & -5 & -1 & -1 & -6 \end{pmatrix}, \begin{pmatrix} -23/30 & -4/15 & -1/10 & 1/6 & 1/6 & 1/5 \\ -4/15 & -23/30 & -1/10 & 1/6 & 1/6 & 1/5 \\ -1/10 & -1/10 & -1/5 & 0 & 0 & 1/5 \\ 1/6 & 1/6 & 0 & -1/3 & 0 & 0 \\ 1/6 & 1/6 & 0 & 0 & -1/3 & 0 \\ 1/5 & 1/5 & 1/5 & 0 & 0 & -2/5 \end{pmatrix}.$$

The discriminant group (cf. **(2.0)**, $A_{L^{G_N}} \cong$) $A_{N(Rt)^{G_N}} = \mathrm{Hom}(N(Rt)^{G_N}, \mathbf{Z})/N(Rt)^{G_N}$ $\cong \mathbf{Z}/(30) \oplus \mathbf{Z}/(30)$ and is generated by cosets $\overline{e}_1^*$ and $\overline{e}_2^* + \overline{e}_3^* + \overline{e}_4^*$ with intersection form:

$$\begin{pmatrix} (\overline{e}_1^*)^2 & \overline{e}_1^*.(\overline{e}_2^* + \overline{e}_3^* + \overline{e}_4^*) \\ \overline{e}_1^*.(\overline{e}_2^* + \overline{e}_3^* + \overline{e}_4^*) & (\overline{e}_2^* + \overline{e}_3^* + \overline{e}_4^*)^2 \end{pmatrix} = \begin{pmatrix} -23/30 & -1/5 \\ -1/5 & -35/30 \end{pmatrix}.$$

(4) For Case(2ii), the intersection matrix $M_2 = (e_i.e_j)$ and its inverse M_2^{-1} are respectively:

$$\begin{pmatrix} -2 & 0 & 0 & -1 & -1 & -1 \\ 0 & -2 & 0 & -1 & -1 & -1 \\ 0 & 0 & -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & -4 & -1 & -1 \\ -1 & -1 & 0 & -1 & -4 & -1 \\ -1 & -1 & 0 & -1 & -1 & -6 \end{pmatrix}, \begin{pmatrix} -11/15 & -7/30 & -1/10 & 1/5 & 1/6 & 1/10 \\ -7/30 & -11/15 & -1/10 & 1/5 & 1/6 & 1/10 \\ -1/10 & -1/10 & -3/5 & 1/5 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & -2/5 & 0 & 0 \\ 1/6 & 1/6 & 0 & 0 & -1/3 & 0 \\ 1/10 & 1/10 & 0 & 0 & 0 & -1/5 \end{pmatrix}.$$

The discriminant group $A_{N(Rt)^{G_N}}$ is isomorphic to $\mathbf{Z}/(30) \oplus \mathbf{Z}/(10)$ and generated by the cosets $\overline{e}_1^*$ and $\overline{e}_3^*$ with intersection form:

$$\begin{pmatrix} (\overline{e}_1^*)^2 & \overline{e}_1^*.\overline{e}_3^* \\ \overline{e}_1^*.\overline{e}_3^* & (\overline{e}_3^*)^2 \end{pmatrix} = \begin{pmatrix} -11/15 & -1/10 \\ -1/10 & -3/5 \end{pmatrix}.$$

(5) For Case(2iii), the intersection matrix $M_3 = (e_i.e_j)$ and M_3^{-1} are respectively:

$$\begin{pmatrix} -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 & 0 & -1 \\ -1 & -1 & -1 & 0 & -4 & 0 \\ 0 & 0 & 0 & -1 & 0 & -8 \end{pmatrix}, \begin{pmatrix} -3/5 & -1/10 & -1/10 & 0 & 1/5 & 0 \\ -1/10 & -3/5 & -1/10 & 0 & 1/5 & 0 \\ -1/10 & -1/10 & -3/5 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & -8/15 & 0 & 1/15 \\ 1/5 & 1/5 & 1/5 & 0 & -2/5 & 0 \\ 0 & 0 & 0 & 1/15 & 0 & -2/15 \end{pmatrix}.$$

The discriminant group $A_{N(Rt)^{G_N}}$ is isomorphic to $\mathbf{Z}/(30) \oplus \mathbf{Z}/(10)$ and generated by the cosets $\overline{e}_2^*$ and $\overline{e}_1^* + \overline{e}_4^*$ with intersection form:

$$\begin{pmatrix} (\overline{e}_2^*)^2 & \overline{e}_2^*.(\overline{e}_1^* + \overline{e}_4^*) \\ \overline{e}_2^*.(\overline{e}_1^* + \overline{e}_4^*) & (\overline{e}_1^* + \overline{e}_4^*)^2 \end{pmatrix} = \begin{pmatrix} -3/5 & -1/10 \\ -1/10 & 13/15 \end{pmatrix}.$$

(6) For Case(2iv), the intersection matrix $M_4 = (e_i.e_j)$ and M_4^{-1} are respectively:

$$\begin{pmatrix} -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & 0 & -1 \\ -1 & -1 & 0 & 0 & -6 & 0 \\ 0 & 0 & -1 & -1 & 0 & -6 \end{pmatrix}, \begin{pmatrix} -11/20 & -1/20 & 0 & 0 & 1/10 & 0 \\ -1/20 & -11/20 & 0 & 0 & 1/10 & 0] \\ 0 & 0 & -11/20 & -1/20 & 0 & 1/10 \\ 0 & 0 & -1/20 & -11/20 & 0 & 1/10 \\ 1/10 & 1/10 & 0 & 0 & -1/5 & 0 \\ 0 & 0 & 1/10 & 1/10 & 0 & -1/5 \end{pmatrix}.$$

The discriminant group $A_{N(Rt)^{G_N}}$ is isomorphic to $\mathbf{Z}/(20) \oplus \mathbf{Z}/(20)$ and generated by the cosets $\overline{e}_1^*$ and $\overline{e}_3^*$ with intersection form:

$$\begin{pmatrix} (\overline{e}_1^*)^2 & \overline{e}_1^*.\overline{e}_3^* \\ \overline{e}_1^*.\overline{e}_3^* & (\overline{e}_3^*)^2 \end{pmatrix} = \begin{pmatrix} -11/20 & 0 \\ 0 & -11/20 \end{pmatrix}.$$

(7) For Case(2v), the intersection matrix $M_5 = (e_i.e_j)$ and its inverse M_5^{-1} are respectively:

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & -20 & 0 \\ 0 & -1 & 0 & -1 & 0 & -8 \end{pmatrix}, \begin{pmatrix} -41/60 & -11/30 & -1/60 & -1/30 & 0 & 1/20 \\ -11/30 & -11/15 & -1/30 & -1/15 & 0 & 1/10 \\ -1/60 & -1/30 & -41/60 & -11/30 & 0 & 1/20 \\ -1/30 & -1/15 & -11/30 & -11/15 & 0 & 1/10 \\ 0 & 0 & 0 & 0 & -1/20 & 0 \\ 1/20 & 1/10 & 1/20 & 1/10 & 0 & -3/20 \end{pmatrix}.$$

The discriminant group $A_{N(Rt)^{G_N}}$ is isomorphic to $\mathbf{Z}/(60) \oplus \mathbf{Z}/(20)$ and generated by the cosets $\overline{e}_1^*$ and $\overline{e}_5^*$ with intersection form:

$$\begin{pmatrix} (\overline{e}_1^*)^2 & \overline{e}_1^*.\overline{e}_5^* \\ \overline{e}_1^*.\overline{e}_5^* & (\overline{e}_5^*)^2 \end{pmatrix} = \begin{pmatrix} -41/60 & 0 \\ 0 & -1/20 \end{pmatrix}.$$

(8) For Case(2vi), The intersection matrix $M_6 = (e_i.e_j)$ and M_6^{-1} are respectively:

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -12 & 0 & 0 \\ 0 & -1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix}, \begin{pmatrix} -7/10 & -2/5 & 0 & 0 & 1/10 & 0 \\ -2/5 & -4/5 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & -1/10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/12 & 0 & 0 \\ 1/10 & 1/5 & 0 & 0 & -3/10 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/4 \end{pmatrix}.$$

The discriminant group $A_{N(Rt)^{G_N}} = \mathbf{Z}/(60) \oplus \mathbf{Z}/(20) \oplus \mathbf{Z}/(2) \oplus \mathbf{Z}/(2) = \mathbf{Z}/(10) \oplus \mathbf{Z}/(10) \oplus \mathbf{Z}/(12) \oplus \mathbf{Z}/(4)$ and the latter is generated by the cosets $\overline{e}_j^*$ $(j = 1, 3, 4, 6)$.

(9) In both of the cases of M_2 and M_3, the discriminant group $A_{N(Rt)^{G_N}}$ is isomorphic to the group $\langle \overline{t}_1^*, \overline{t}_2^* \rangle \cong \mathbf{Z}/(30) \oplus \mathbf{Z}/(10)$ with the intersection matrix $(\overline{t}_i^*.\overline{t}_j^*) = \begin{pmatrix} 1/15 & 1/30 \\ 1/30 & 1/15 \end{pmatrix}$.

We now prove **(2.1)**. Since $\operatorname{rank} N(Rt)^{G_N} = 6$, the G_N-action on the 24 simple roots of Rt has exactly 6 orbits.

We argue as in the proof of [Ko1, Theorem 4]. The fact that $G_N = A_5 < S(N(Rt))$ implies that Rt is one of the following: $24A_1, 12A_2, 6A_4, 6D_4$.

If $Rt = 6A_4$, then $S(N(Rt)) = 2.PGL_2(5)$ ($< 2.S_6$) [CS, Ch 16, §1], where the order 2 element acts as a symmetry of order 2 on each connected component of Dynkin type A_4, and $PGL_2(5)$ acts on the set (identified with $\{0, 1, 2, 3, 4, \infty\}$) of 6 components of Rt as permutations in a natural way. Since A_5 is simple, the composition of homomorphisms below is an injection: $A_5 \subset S(N(Rt)) \to PGL_2(5)$, so we may assume that $A_5 < PGL_2(5)$. Since $G_N = A_5$ fixes one simple root of Rt by the construction, our A_5 is a subgroup of the stabilizer subgroup of $PGL_2(5)$ and this stablizer is of order $|PGL_2(5)|/6 = 20$. This is impossible because $|A_5| = 60 > 20$.

If $Rt = 6D_4$, then $S(N(Rt)) = 3.S_6$, where the order 3 element acts as a symmetry of order 3 on each connected component of Dynkin type D_4, and S_6 acts on the set of 6 connected components of Rt as permutations. As above, the simplicity of G_N implies that the subgroup G_N of $S(N(Rt))$ is indeed a subgroup of S_6. Since $G_N = A_5$ fixes one simple root of Rt, our group A_5 is a subgroup ($= [S_5, S_5]$) of the stabilizer subgroup S_5 of S_6. So this A_5 acts transitively on the remaining 5 connected components of Rt and hence the G_N-action on the 24 simple roots has exactly 8 orbits, noting that one connected component of Rt is component wise fixed by G_N, a contradiction.

Suppose that $Rt = 12A_2$. Then $S(N(Rt)) = 2.M_{12}$, where the order 2 element acts as a symmetry of order 2 on each connected component of Dynkin type A_2, and the Mathieu group M_{12} acts on the set of 12 connected components of Rt as permutations. Let $r_{2k-1} + r_{2k}$ $(1 \le k \le 12)$ be the 12 connected components of Rt with r_j the 24 simple roots. Every non-trivial element of $N(Rt)/Rt$ is of the form $\sum_{i \in H} \pm(r_{2i-1} + 2r_{2i})/3$ where H is an element of the ternary Golay code and $|H| = 6, 9, 12$ [CS, Ch 3, §2.8.5]. Since the group $G_N = A_5$ is simple and fixes one simple root of Rt, this G_N is a subgroup of M_{12} and indeed, a subgroup of the stabilizer subgroup M_{11} of M_{12}. Suppose the G_N-orbit decomposition on the 12 connected components is $1 + a + b$. Then the G_N-orbit decomposition of the 24 simple roots is $[1, 1, a, a, b, b]$ (so $a + b = 11$), where aA_2 (resp.

bA_2) is split into two G_N-orbits with a (resp. b) disjoint simple roots each. Thus Case (2v) or (2vi) occurs by **(1.7)**.

Suppose that $Rt = 24A_1$. Then $S(N(Rt)) = M_{24}$. The elements of $N(Rt)/Rt$ form the binary Golay code. Since $G_N = A_5$ fixes one simple root of Rt, our group G_N is the stabilizer subgroup M_{23} of M_{24}. Let the G_N-orbit decomposition of the 24 simple roots be $[1, a, b, c, d, e]$ with $a \le b \le c \le d \le e$ (so $a+b+c+d+e = 23$). By **(1.7)**, all a, b, c, d, e are in $\{1, 5, 6, 10, 12, 15, 20\}$ and hence Cases (2i) - (2vi) occur.

(4) According to the ordering of $[1, 1, 1, 5, 6, 10]$, we label the orbits as $O_1 = \{r_1\}, O_1' = \{r_2\}, O_1'' = \{r_3\}, O_5 = \{r_4, \dots, r_8\}, O_6 = \{r_9, \dots, r_{14}\}, O_{10} = \{r_{15}, \dots, r_{24}\}$, where r_j are the 24 simple roots. We claim that $O_1 + O_1' + O_1'' + O_5$ (to be precise, after divided by 2) is an octad, and $O_1 + O_1' + O_6$ is also an octad (after relabelling O_1, O_1', O_1''). So

$$e_i = r_i (1 \le i \le 3),\ e_4 = \frac{1}{2}(O_1 + O_1' + O_1'' + O_5),\ e_5 = \frac{1}{2}(O_1 + O_1' + O_6),\ e_6 = \frac{1}{2}(O_1 + O_1' + O_{10})$$

form a basis of $N(Rt)^{G_N}$, noting that the last dodecad is the complement of the symmetric sum (a dodecad) of the two octads above and that except for the above-mentioned two octads and two dodecads, there is no any other octad or dodecad which is a union of orbits. Indeed, let Oct_1 be the unique octad containing O_5. Note that the cycle type in M_{24} of an order-5 element γ in A_5 is (5^4) , Appendix B, Table 5.I]. So γ is of type (5^2) (resp. (5)) on O_{10} (resp. on O_5 and O_6). Since $\gamma(Oct_1) \cap Oct_1$ contains O_5, we have $\gamma(Oct_1) = Oct_1$. If Oct_1 contains an element of O_{10} then it contains the five images in O_{10} by the action of $\langle \gamma \rangle$, so $|Oct_1| \ge 10$, absurd. If Oct_1 contains an element r_j in O_6 we may choose γ not fixing r_j (note that the stabilizer subgroup of A_5, regarded as a subgroup of $\mathrm{Sym}(O_6) = S_6$ and fixing an element ($\ne r_j$) in O_6, has order 10 and hence gives rise to such γ). Then we will get a similar contradiction. Thus $Oct_1 = O_1 + O_1' + O_1'' + O_5$ as claimed.

Let Oct_2 be the unique octad containing the first 5 elements in O_6. Let γ be an order-5 element in A_5 fixing the last element in O_6. Then $\gamma(Oct_2) = Oct_2$. As above, this implies that Oct_2 is disjoint from O_5 and O_{10}. So either $Oct_2 = O_1 + O_1' + O_6$ after relabelling the 1-element orbits, or Oct_2 is the union of the 5 elements in O_6 and the three 1-element orbits (this leads to that the symmetric sum of Oct_1 and Oct_2 is a 10-word Golay code, absurd).

(3) For the orbit decomposition $[1,1,5,5,6,6]$, we label the orbits as $O_1 = \{r_1\}$, $O_1' = \{r_2\}$, $O_5 = \{r_3, \ldots, r_7\}$, $O_5' = \{r_8, \ldots, r_{12}\}$, $O_6 = \{r_{13}, \ldots, r_{18}\}$, $O_6' = \{r_{19}, \ldots, r_{24}\}$. As in (4), we can prove that both $O_1 + O_1' + O_6$ and $O_1 + O_1' + O_6'$ are octads. Thus $N(Rt)^{G_N}$ has a basis below, noting that except for the two octads, the symmetric sum (a dodecad) of the two octads and the complement (another dodecad) of this dodecad, there is no other octad or dodecad which is the union of orbits:

$$e_i = r_i (i=1,2),\ e_3 = O_5,\ e_4 = \frac{1}{2}(O_1 + O_1' + O_6),\ e_5 = \frac{1}{2}(O_1 + O_1' + O_6'),\ e_6 = \frac{1}{2}(O_1 + O_1' + O_5 + O_5')$$

.

(5) For the orbit decomposition $[1,1,1,1,5,15]$, we label the orbits as $O_1 = \{r_1\}$, $O_1' = \{r_2\}$, $O_1'' = \{r_3\}$, $O_1''' = \{r_4\}$, $O_5 = \{r_5, \ldots, r_9\}$, $O_{15} = \{r_{10}, \ldots, r_{24}\}$. As in (4), we may assume that $O_1 + O_1' + O_1'' + O_5$ is an octad after relabelling the 1-element orbits and that there is no any other octad or dodecad which is a union of orbits. Thus $N(Rt)^{G_N}$ has a basis:

$$e_i = r_i (i = 1,2,3,4),\ e_5 = \frac{1}{2}(O_1 + O_1' + O_1'' + O_5),\ e_6 = \frac{1}{2}(O_1''' + O_{15}).$$

(6) For the orbit decomposition $[1,1,1,1,10,10]$, we label the orbits as $O_1 = \{r_1\}$, $O_1' = \{r_2\}$, $O_1'' = \{r_3\}$, $O_1''' = \{r_4\}$, $O_{10} = \{r_5, \ldots, r_{14}\}$, $O_{10}' = \{r_{15}, \ldots, r_{24}\}$. Take an order-5 element γ of A_5. So O_{10} splits into two 5-element subsets on each of which γ acts transitively. Let Oct_j $(j = 1,2)$ be the unique octad containing the first (resp. second) 5-element subset. As in (4), we can show that each Oct_j is the union of the 5-element subset and three 1-element orbits. The symmetric sum of Oct_1 and Oct_2 is a dodecad which may be assumed to be $O_1 + O_1' + O_{10}$; its complement is also a dodecad. Except for these two dodecads, there is no any other dodecad which is a union of orbits. Thus $N(Rt)^{G_N}$ has a basis:

$$e_i = r_i (i = 1,2,3,4),\ e_5 = \frac{1}{2}(O_1 + O_1' + O_{10}),\ e_6 = \frac{1}{2}(O_1'' + O_1''' + O_{10}').$$

(8) For $Rt = 12A_2$ and the orbit decomposition $[1,1,5,5,6,6]$, we label the orbits as $O_1 = \{r_1\}$, $O_1' = \{r_2\}$, $O_5 = \{r_3, r_5, \ldots, r_{11}\}$, $O_5' = \{r_4, r_6, \ldots, r_{12}\}$, $O_6 = \{r_{13}, r_{15}, \ldots, r_{23}\}$,

$O_6' = \{r_{14}, r_{16}, \dots, r_{24}\}$, where $r_{2k-1} + r_{2k}$ $(1 \le k \le 12)$ are the 12 connected components of Rt. Every non-trivial element of the group $N(Rt)/Rt$ is represented by some $\gamma_H = \sum_{i \in H} \pm(r_{2i-1} + 2r_{2i})/3$ where H is an element of the ternary Golay code (so $|H| = 6, 9, 12$) which is also the Steiner system $St(5, 6, 12)$ [Atlas]. Let H_i with $i = 1$ (resp. $i = 2$) be the unique element of the ternary Golay code with $|H_i| = 6$ such that $\gamma_{H_1} = \frac{1}{3}\sum_{i=2}^{6} \pm(r_{2i-1} + 2r_{2i}) \pm \frac{1}{3}(r_{2j_1-1} + 2r_{2j_1})$ for some j_1 (resp. $\gamma_{H_2} = \frac{1}{3}\sum_{i=7}^{11} \pm(r_{2i-1} + 2r_{2i}) \pm \frac{1}{3}(r_{2j_2-1} + 2r_{2j_2})$ for some j_2); such Golay code can also be constructed from the binary Golay code = Steiner system $St(5, 8, 24)$ where such H_i is the intersection of a fixed dodecad and an octad. Using the fact that an order-5 element in A_5 has cycle type (5^2) in M_{12} [EDM], as in the case of $Rt = 24A_1$, we can prove that $N(Rt)^{G_N}$ has a basis:

$$e_i = r_i (i = 1, 2),\ e_3 = \sum_{k=2}^{6} r_{2k-1},\ e_4 = \sum_{k=7}^{12} r_{2k-1},\ e_5 = \frac{1}{3}\sum_{k=1}^{6}(r_{2k-1}+2r_{2k}),\ e_6 = \frac{1}{3}\sum_{k=7}^{12}(r_{2k-1}+2r_{2k}).$$

(7) For $Rt = 12A_2$ and the orbit decomposition $[1, 1, 1, 1, 10, 10]$, we label the orbits as $O_1 = \{r_1\}$, $O_1' = \{r_2\}$, $O_1'' = \{r_3\}$, $O_1''' = \{r_4\}$, $O_{10} = \{r_5, r_7, \dots, r_{23}\}$, $O_{10}' = \{r_6, r_8, \dots, r_{24}\}$, where $r_{2k-1} + r_{2k}$ $(1 \le k \le 12)$ are the 12 connected components of Rt. As in (8), we can prove that $N(Rt)^{G_N}$ has a basis:

$$e_i = r_i\, (1 \le i \le 4),\ e_5 = \sum_{k=3}^{12} r_{2k-1},\ e_6 = \frac{1}{3}\sum_{k=1}^{12}(r_{2k-1} + 2r_{2k}).$$

(9) follows from the direct calculation. Indeed, in the case of M_2, the isomorphism $\varphi_2 : \langle \bar{t}_1^*, \bar{t}_2^* \rangle \to A_{N(Rt)^{G_N}}$ is given by $(\varphi_2(\bar{t}_1^*), \varphi_2(\bar{t}_2^*)) = (\bar{e}_1^*, \bar{e}_3^*)\begin{pmatrix} 2 & 7 \\ 1 & 0 \end{pmatrix}$. In the case of M_3, the isomorphism $\varphi_3 : \langle \bar{t}_1^*, \bar{t}_2^* \rangle \to A_{N(Rt)^{G_N}}$ is given by $(\varphi_3(\bar{t}_1^*), \varphi_3(\bar{t}_2^*)) = (\bar{e}_2^*, \bar{e}_1^* + \bar{e}_4^*)\begin{pmatrix} 1 & 7 \\ 1 & -4 \end{pmatrix}$. This proves **(2.1)**.

§3. The proof of Theorems A and B

In this section, we shall prove Theorems A and B. We prove first the result below which includes Theorem A.

Theorem 3.1.

(1) There is no faithful group action of the form $A_5.\mu_3$ (see **(1.0)**) on a $K3$ surface.

(2) If $G = A_5.\mu_I$ acts faithfully on a $K3$ surface. Then $G = A_5 : \mu_I$ and $I = 1$, 2, or 4. (It is proved in [Z2] that $I = 4$ is impossible.)

(2) is a consequence of (1) and **(1.1)**. Indeed, if $I = 6$, then the subgroup $H = \alpha^{-1}(\mu_3)$ of $G = A_5.\mu_6$ is of the form $H = A_5.\mu_3$ which is impossible by (1). To prove (1), we need the following result first.

Lemma 3.2. Suppose that $G = A_5.\mu_3$ acts on a $K3$ surface X. Let $\zeta_3 = \exp(2\pi\sqrt{-1}/3)$.

(1) We have $G = A_5 \times \mu_3$. Moreover, a generator g of μ_3 can be chosen so that $g^*|S_X \otimes \mathbf{C} = \mathrm{diag}[1, 1, \zeta_3 I_4, \zeta_3^{-1} I_4, \zeta_3 I_5, \zeta_3^{-1} I_5]$, where the decompositoin here is compatible with that in **(1.6)** in the sense that $g^*|\chi_4 \oplus \chi_4' = \mathrm{diag}[\zeta_3 I_4, \zeta_3^{-1} I_4]$ and $g^*|\chi_5 \oplus \chi_5' = \mathrm{diag}[\zeta_3 I_5, \zeta_3^{-1} I_5]$. In particular, $\chi_{\mathrm{top}}(X^g) = -6$.

(2) We have $S_X^G = S_X^g = S_X^{A_5} = H^0(X, \mathbf{Z})^g$. This lattice is of rank 2 (whose $\mathbf{C}$-extension is $\chi_1 \oplus \chi_1'$) and its discriminant group is 3-elementary.

(3) We have $S_X^{A_5} = U = U(1)$, or $U(3)$, where $U(n) = \mathbf{Z}[u_1, u_2]$ is a rank 2 lattice with $u_i^2 = 0$ and $u_1.u_2 = n$.

Proof. (1) The first part is from **(1.1)**. For a generator g of μ_3, since $o(g) = 3$ and by the form of the decomposition in **(1.6)**, each χ_i ($i = 4, 5$) is g-stable. Since the order-3 element g acts on the rank-2 lattice $S_X^{A_5}$ (which is defined over $\mathbf{Z}$ and whose $\mathbf{C}$-extension is $\chi_1 \oplus \chi_1'$), it has at least one eigenvalue equal to 1 because $G = \langle A_5, g\rangle$ stabilizes an ample line bundle (the pull back of an ample line bundle on X/G). So $g^*|S_X^{A_5} = \mathrm{id}$. The commutativity of g with all elements in A_5 implies that $g^*|\chi_i$ is a scalar multiple, by Schur's lemma.

Thus we can write $g^*|S_X \otimes \mathbf{C} = \mathrm{diag}[1, 1, \zeta_3^b I_4, \zeta_3^c I_4, \zeta_3^d I_5, \zeta_3^e I_5]$, where the ordering is the same as in **(1.6)**. Let $a \in A_5$. Then $(ga)^*|T_X = g^*|T_X$ and the latter can be diagonalized as $\mathrm{diag}[\zeta_3, \zeta_3^{-1}]$, noting that $\mathrm{rank}\, T_X = 22 - \mathrm{rank}\, S_X = 2$ [Ni1, Theorem 0.1], **(1.0A-B)**. So $\mathrm{Tr}(ga)^*|T_X = -1$. As in the proof of **(1.8)**, the topological Lefschetz fixed point formula implies that $\chi(X^{ga}) = 2 + \mathrm{Tr}(ga)^*|T_X + \mathrm{Tr}(ga)^*|S_X = 1 + \mathrm{Tr}(ga)^*|S_X = 3 + \zeta^b \mathrm{Tr}(a^*|\chi_4) + \zeta^c \mathrm{Tr}(a^*|\chi_4') + \zeta^d \mathrm{Tr}(a^*|\chi_5) + \zeta^e \mathrm{Tr}(a^*|\chi_5')$. So for $a = \mathrm{id}$, $2A$, $3A$, $5A$ with

nA denoting an element of order n in A_5, we have:

$$\chi_{\text{top}}(X^g) = 3 + 4(\zeta_3^b + \zeta_3^c) + 5(\zeta_3^d + \zeta_3^e),$$
$$\chi_{\text{top}}(X^{g2A}) = 3 + \zeta_3^d + \zeta_3^e,$$
$$\chi_{\text{top}}(X^{g3A}) = 3 + \zeta_3^b + \zeta_3^c - \zeta_3^d - \zeta_3^e,$$
$$\chi_{\text{top}}(X^{g5A}) = 3 - \zeta_3^b - \zeta_3^c.$$

The fact $\chi(X^{g5A}) = 4$ in **(1.4)** implies that $(\zeta_3^b, \zeta_3^c) = (\zeta_3, \zeta_3^{-1})$ after switching χ_4 with χ_4' if necessary. Since $\chi(X^{g3A}) = 0$ is in $\mathbf{R}$ (in $\mathbf{Z}$, indeed), we may assume that $(\zeta_3^d, \zeta_3^e) = (\zeta_3, \zeta_3^{-1})$, or $(1, 1)$. If the former case occurs then the lemma is true.

Suppose that the latter case occurs. Then $\chi_{\text{top}}(X^g) = 9$, whence $n_g = 2$ and $|X^g_{\text{isol}}| = m_g = n_g + 3 = 5$ by **(1.4)**. Since g commutes with every element in A_5, our A_5 acts on the 5-point set X^g_{isol}. By **(1.7)**, A_5 either fixes a point P_1 of the set (and hence $A_5 < SL(T_{X,P_1})$, contradicting **(1.0C)**), or acts transitively as a subgroup $(= [S_5, S_5])$ of S_5, on the set with an order-12 stabilizer (of a point P_1) subgroup $A_4 < A_5$, so $A_4 < SL(T_{X,P_1})$, contradicting **(1.0C)**. This proves the assertion (1).

(2) The first part follows from (1), that $g^*|T_X \otimes \mathbf{C} = \text{diag}[\zeta_3, \zeta_3^{-1}]$ w.r.t. to a suitable basis by [Ni1, Theorem 0.1] and that all lattices in (2) are primitive (of the same rank as they turn out to be) in $L := H^2(X, \mathbf{Z})$. We still have to show that the discriminant group $A_{L^g} = \text{Hom}(L^g, \mathbf{Z})/L^g$ of L^g is 3-elementary. Let $L_g = (L^g)^\perp$ be the orthogonal of L^g in L. Then $g^*|L_g$ has only $\zeta_3^\pm$ as eigenvalues. Now arguing as in [OZ2, Lemma (1.3)] (for the finite index sublattice $L^g \oplus L_g$ of L, instead of $S_X \oplus T_X$), we can show that A_{L^g} is 3-elementary.

(3) follows from (2). See [CS, Table 15.2a].

The fixed locus X^g can be determined:

Lemma 3.3. (1) With the assumption and notation in **(3.1)** and **(3.2)**, either $X^g = C \coprod R$ is a disjoint union of a genus-5 curve C and a curve R ($\cong \mathbf{P}^1$) (so $C^2 = 8$, and $S_X^g = U \supset \mathbf{Z}[C, R]$), or X^g equals a sinlge genus-4 curve C (so $C^2 = 6$).

(2) In the former case, $\Phi_{|C|} : X \to \mathbf{P}^5$ is a degree-2 morphism onto either the Veronese-embedded $\mathbf{P}^2$ in $\mathbf{P}^5$ or the normal cone $\overline{\Sigma}_4$ over a rational normal twisted quartic in $\mathbf{P}^4$.

Proof. Since $\chi(X^g) = -6$ by **(3.2)**, we have $n_g = -3$ and $m_g = 0$ in notation of **(1.4)**. $n(g) < 0$ infers that X^g is a disjoint union of a smooth curve C of genus ≥ 2 and t of $\mathbf{P}^1$'s with $-6 = 2 - 2g(C) + 2t$ (see **(1.2)**). The fact that $\operatorname{rank} S_X^g = 2$ in **(3.2)** implies that either $t = 0$ (so $g(C) = 4$), or $t = 1$ (so $g(C) = 5$) so that the two curves in X^g give rise to two linearly independent classes of S_X^g.

If $S_X^g = U(3)$, then $C^2 = 0 \pmod 3$ because C is in S_X^g, whence $C^2 = 6$. This proves the first assertion of the lemma, by virtue of **(3.2)**.

Consider the caser $X^g = C \coprod R$. By [SD, Theorem 3.1], $|C|$ is base point free and we have a morphism $\varphi := \Phi_{|C|} : X \to \mathbf{P}^5$. Now $8 = C^2 = \deg(\varphi).\deg(\operatorname{Im}\varphi)$, where $\deg(\operatorname{Im}\varphi) \geq 5-1$. Thus either φ is an embedding modulo the curves in $C^\perp$, or φ is a degree-2 map as described in **(3.3)** [SD, Theorem 5.2, Propositions 5.6 and 5.7].

Write $S_X^g = \mathbf{Z}[u_1, u_2]$ with $u_i^2 = 0$ and $u_1.u_2 = 1$. Express $C = a_1u_1 + a_2u_2$. Then $8 = 2a_1a_2$ and we may assume that $(a_1, a_2) = (2, 2)$ or $(4, 1)$ (after replacing u_i by $-u_i$ or switching u_1 with u_2 if necessary). So $C.u_i > 0$ and hence the Riemann-Roch theorem implies that $\dim |u_1| \geq 1$. Write $|u_1| = |M| + F$ with $|M|$ the movable part. Then $0 < C.M \leq C.u_1 = a_2 \leq 2$. If φ is birational then $\varphi(M)$ is a plane conic or a line, whence $M \cong \mathbf{P}^1$, $M^2 = -2$ and $|M|$ is not movable, a contradiction. This proves the lemma.

We now prove **(3.1)** (1). Consider the case in **(3.3)**, where $X^g = C \coprod R$ and $\varphi = \Phi_{|C|} : X \to \mathbf{P}^5$ is a degree-2 morphism onto the Veronese-embedded $\mathbf{P}^2$ in $\mathbf{P}^5$. Since C (and hence $|C|$) is G-stable, there is an induced action of G on $\mathbf{P}^5$ (and hence also an action of G on the image $\varphi(X) = \mathbf{P}^2$) so that the map φ is G-equivariant. The $G = A_5 \times \mu_3$ action on the image is also faithful because A_5 is simple and $\deg(\varphi) = 2$ is coprime to 3 $(= |\mu_3|)$. The action of A_5 on the image is via $A_5 \subset SL_3(\mathbf{C}) \subset PGL_3(\mathbf{C})$ and is given in Burnside [Bu, §232, or §266] **(1.8).** In particular, the commutativity of g with the two generators (order 5 and 2) of A_5 in [Bu, §266] shows that g is a scalar and acts trivially on the image $\varphi(X) = \mathbf{P}^2$, a contradiction.

Consider the case in **(3.3)**, where $X^g = C \coprod R$ in **(3.3)** and $\varphi = \Phi_{|C|} : X \to \mathbf{P}^5$ is a degree-2 morphism onto the cone $\overline{\Sigma}_4$. Note that the minimal resolution Σ_4 of $\overline{\Sigma}_4$ is the Hirzebruch surface of degree 4. As in the previous case, there is a faithful action of G on $\overline{\Sigma}_4$ such that φ is G-equivariant. Note that the image $\varphi(C)$ is a hyperplane section away from the singularity and with $\varphi(C)^2 = 4$. Let ℓ be a generating line of the cone $\overline{\Sigma}_4$. Then $\varphi(C) \sim 4\ell$ as Weil divisors. This gives rise to a $\mathbf{Z}/(4)$-cover $\pi : Y = Spec \oplus_{i=0}^3 \mathcal{O}_{\overline{\Sigma}_4}(-i\ell) \to \overline{\Sigma}_4$ which is (totally) ramified exactly over $\varphi(C)$. One sees that $Y \cong \mathbf{P}^2$ and $\pi^*\varphi(C) = 4L$ with L a line. Clearly, A_5 ($< G$) stabilizes the divisorial sheaves $\mathcal{O}(-i\ell)$ and fixes the defining equation of $\varphi(C)$, so there is an induced faithful A_5-action on $Y = \mathbf{P}^2$ so that π is A_5-equivariant (see **(1.7)**). Now L is stabilized by A_5 (because so is $\varphi(C)$). So the defining equation $F_1 = 0$ of L is semi A_5-invariant (and hence A_5-invariant because of the simplicity of the group A_5). But every A_5-invariant form is of even degree by [Bu, §266], noting also that the action of A_5 on $\mathbf{P}^2$ is via $A_5 \subset SL_3(\mathbf{C}) \to PGL_3(\mathbf{C})$ by **(1.8)**. We reach a contradiction.

Consider the case $X^g = C$ in **(3.3)**. Let $f : X \to Y = X/\langle g \rangle$ be the quotient map. There is an induced faithful action A_5 on Y so that f is A_5-equivariant. Then by the ramification divisor formula, $0 \sim K_X = f^*(K_Y) + 2C$. Pushing down by f_*, one obtains $0 \sim 3K_Y + 2B$ with $B = f_*C = f(C) \cong C$ and $f^*B = 3C$, so $B^2 = 3C^2 = 18$. Solving, one obtains $B = (-3/2)K_Y$ and $K_Y^2 = 8$. Thus the smooth ruled surface Y equals a Hirzebruch surface Σ_d of degree d. The irreducibility of B (being a $\mathbf{Z}$-divisor) implies that $d = 0, 2$ [Ha, Ch V, Cor. 2.18].

Suppose that $d = 2$. Then the (-2)-curve M on Y is disjoint from $B = (-3/2)K_Y$ and hence $f^*M = \coprod_{i=1}^3 M_i$ is a disjoint union of three (-2)-curves not intersecting C. Since M is clearly A_5-stable, the set $\coprod M_i$ is also A_5-stable, whence each M_i is A_5-stable **(1.7)**. An order-5 element $5A$ in A_5 acts on each M_i faithfully by **(1.2)** and has exactly two fixed points by **(1.8)**. But according to **(1.2)**, $4 = |X^{5A}| \ge \sum_{i=1}^3 |M_i^{5A}| = 6$, a contradiction.

Thus $d = 0$. Clearly, the simple group A_5 stabilizes each ruling and there is an induced action $\rho_i : A_5 \times \mathbf{P}^1 \to \mathbf{P}^1$ with $i = 1, 2$ for the i-th $\mathbf{P}^1$ in $Y = \Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$ so that $\rho_1 \times \rho_2$ is the given A_5 action on Y. Changing coordinates suitably we may assume that the A_5 action on Y commutes with the involution ι of Y switching the two $\mathbf{P}^1$'s in Y, i.e.,

$\rho_1 = \rho_2$ as actions of A_5 on the same $\mathbf{P}^1$ **(1.8)**.

Let $j : Y \to Z = Y/\langle\iota\rangle = \mathbf{P}^2$ be the quotient map. Then there is an induced faithful action of A_5 on $\mathbf{P}^2$ such that j is A_5-equivariant. Now $\iota(B)$ is an irreducible curve with $(\iota(B))^2 = B^2/2 = 9$. It is a cubic curve and A_5-stable because so is $B = f(C)$. The action of A_5 on $Y/\langle\iota\rangle = \mathbf{P}^2$ is via $SL_3(\mathbf{C}) \to PGL_3(\mathbf{C})$ **(1.8)**. The defining equation F_3 of $\iota(B)$ is then a cubic form and semi A_5-invariant (and hence A_5-invariant by the simplicity of the group A_5). However, Burnside [Bu, §266] shows that every A_5-invariant form is of even degree, a contradiction. This completes the proof of **(3.1)** (1) and also of **(3.1)**.

We now prove Theorem B. Suppose that $G = A_5.C_n$ acts faithfully on a $K3$ surface X. By **(1.0A)**, $A_5 \le G_N$. So $G_N = A_5$, S_5, A_6 or $M_{20} = C_2^{\oplus 4} : A_5$ by [Xi, the list]. In notation of **(1.0)**, for some $m \,|\, n$, we have $G_N = \mathrm{Ker}(\alpha) = A_5.C_m$ and $G/G_N = \mu_I$, where $n = mI$. By the same proof of **(1.1)**, we have $I = 1, 2, 3, 4$, or 6. Let $h \in G$ such that the coset of h is a generator of $G/A_5 = C_n$. Then $h^*\omega_X = \eta_I \omega_X$ for some primitive I-th root η_I of 1. Note that $n \,|\, \mathrm{ord}(h)$ and $h^I \in G_N$, whence $\mathrm{ord}(h^I) \le 8$ by **(1.2)**. Thus $\mathrm{ord}(h) = I\,\mathrm{ord}(h^I)$ and $m \,|\, \mathrm{ord}(h^I)$. In particular, $|G_N| = m|A_5| \le 8|A_5|$. Hence $G_N \ne M_{20}$.

If $G_N = A_5.C_m = A_6$, then $m = 6$ and A_6 includes $\langle h^I\rangle \ge C_6$, which is impossible. If $G_N = A_5$, then $C_n = \mu_I$ in notation of **(1.0)**, and Theorem B follows from **(3.1)**.

Consider the case $G_N = S_5$. Then $m = 2$ and $n = 2I$. Moreover, $G_N = \langle A_5, h^I\rangle$. So $h^I \in S_5 - A_5$. Since $S_5 \to \mathrm{Aut}(S_5)$ $(x \mapsto c_x)$ is an isomorphism, we have $c_h = c_s$ for some $s \in S_5$. Set $g = hs^{-1}$. Then g commutes with every element in S_5 and also $\alpha(g) = \alpha(h)$ is a generator of $\mathrm{Im}(\alpha) = \mu_I$. Now $g^I \in \mathrm{Ker}(\alpha) = G_N = S_5$ is in the centre of S_5 (which is (1)). So $\mathrm{ord}(g) = I$ and $G = G_N \times \langle g\rangle = S_5 \times \mu_I > A_5 \times \mu_I$. By **(3.1)**, we have $I = 1, 2$ or 4. The $S_5 \times \mu_I$ should have an element h such that $h^I \in S_5 - A_5$ (i.e., h^I is not an even permutation). Thus, $I \ne 2$, or 4. Therefore, $I = 1$ and $G = G_N = S_5$. This completes the proof of Theorem B.

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